

James' Conjecture for Hecke algebras of exceptional type, I

Meinolf Geck

Department of Mathematical Sciences, King's College, Aberdeen University, AB24 3UE, Scotland, UK

Jürgen Müller

Lehrstuhl D für Mathematik, RWTH Aachen, Templergraben 64, D-52062 Aachen, Germany

Abstract

In this paper, and a second part to follow, we complete the programme (initiated more than 15 years ago) of determining the decomposition numbers and verifying James' Conjecture for Iwahori–Hecke algebras of exceptional type. The new ingredients which allow us to achieve this aim are:

- the fact, recently proved by the first author, that all Hecke algebras of finite type are cellular in the sense of Graham–Lehrer, and
- the explicit determination of W -graphs for the irreducible (generic) representations of Hecke algebras of type E_7 and E_8 by Howlett and Yin.

Thus, we can reduce the problem of computing decomposition numbers to a manageable size where standard techniques, e.g., Parker's **MeatAxe** and its variations, can be applied. In this part, we describe the theoretical foundations for this procedure.

Key words: Hecke algebra, decomposition numbers, James' conjecture

1991 MSC: Primary 20C08

To Gus Lehrer on his 60th birthday

1. Introduction

Let k be a field and q a non-zero element of k . Let $H_n(k, q)$ be the Iwahori–Hecke algebra of type A_{n-1} with parameter q ; this is a certain deformation of the group algebra

Email addresses: `geck@maths.abdn.ac.uk` (Meinolf Geck), `Juergen.Mueller@math.rwth-aachen.de` (Jürgen Müller).

of the symmetric group \mathfrak{S}_n . In order to study the representation theory of $H_n(k, q)$, Dipper and James [5] developed a q -version of the classical theory of Specht modules for \mathfrak{S}_n . In this framework, one obtains a natural parametrization of $\text{Irr}(H_n(k, q))$ (the set of irreducible representations, up to isomorphism) in terms of e -regular partitions, where the parameter e is defined by

$$e = \min\{i \geq 2 \mid 1 + q + q^2 + \cdots + q^{i-1} = 0\}.$$

(We set $e = \infty$ if no such i exists.) If k has characteristic 0, then we also know how to determine the dimensions of the irreducible representations, thanks to the Lascoux–Leclerc–Thibon conjecture [30] and its proof by Ariki [1]. However, the analogous problem for k of positive characteristic is completely open.

Assume now that $e < \infty$ and $\text{char}(k) = \ell > 0$. Based on empirical evidence for $n = 2, 3, \dots, 10$, James [28] made the remarkable conjecture that if $e\ell > n$, then $\text{Irr}(H_n(k, q))$ only depends on e . More precisely, James predicts that $\text{Irr}(H_n(k, q))$ could be obtained from the \mathbb{C} -algebra $H_n(\mathbb{C}, \sqrt[e]{1})$ by a process of ℓ -modular reduction. Shortly afterwards, the first-named author [8] formulated a version of James’ conjecture for Iwahori–Hecke algebras associated to finite Weyl groups in general, and proved that it holds in the so-called “defect 1 case”. (In type A_{n-1} , this corresponds to the case where e divides exactly one of the numbers $2, 3, \dots, n$.) The article [8] also contains an argument which shows that the irreducible representations of any Iwahori–Hecke algebra over a field of characteristic $\ell > 0$ can always be obtained by ℓ -modular reduction from an algebra in characteristic 0, as long as ℓ is large enough. Thus, James’ conjecture and its generalizations are really about finding the correct bound for ℓ .

By ad hoc computational methods, the general version of James’ conjecture has been shown to hold for Iwahori–Hecke algebras of type F_4 and E_6 ; see [18], [9]. These methods, however, turned out to be completely inadequate to deal with algebras of larger rank; in particular, types E_7 and E_8 remained far out of reach.

Using the Kazhdan–Lusztig theory of cells [32] and the Graham–Lehrer concept of abstract “cell data” [23], it was recently shown in [16] that a suitable theory of “Specht modules” exists for Iwahori–Hecke algebras associated to finite Weyl groups in general. First of all, this has the theoretical implication that we can now formulate a general version of James’ conjecture which is, perhaps, more natural than the one in [8]. Furthermore, this has the practical implication of leading to an algorithm for verifying the general version of James’ conjecture, in which the main issue is the determination of the invariant bilinear form (and its rank) on a “cell representation”.

In order to make this work, a number of problems have to be resolved. To begin with, we need explicit models for those “cell representations”. For W of exceptional type, we will see that such models are given by the W -graph representations which were recently obtained by Howlett and Yin [26], [40] and which are readily accessible through Michel’s development version [35] of the computer algebra system CHEVIE [17]. Then the determination of the invariant bilinear form essentially amounts to solving a system of linear equations. This works fine for dimensions of up to around 2500, but some more refined methods are necessary for dealing with the large representations (of dimension up to 7168) in type E_8 . The discussion of these finer computational methods is beyond the scope of the present article and can be found in [19].

Still, with all these new tools at hand, the computations required to determine the Gram matrices of the invariant bilinear forms for large representations in type E_8 takes

several months of CPU time on modern computers. Note, however, that once these matrices have been computed, it is relatively easy to verify that they indeed define invariant bilinear forms and to compute their ranks for various specialisations. It is planned to create a data base which makes these data generally available.

This paper is organised as follows. In Section 2, we recall the construction of “cell data” à la Graham–Lehrer in Iwahori–Hecke algebras associated to finite Weyl groups. We also discuss the example of type G_2 , which provides a first illustration for the phenomenon expressed in James’ conjecture. In Section 3, we formulate the general version of James’ conjecture using the new approach based on cell representations. The equivalent formulation in Corollary 3.6 provides the conceptual basis for the algorithm for verifying James’ conjecture. In Section 4, we discuss the main computational issues in this algorithm and show how they can be solved—at least in principle. In particular, in §4.2, we prove a general result which allows us to verify that the Howlett–Yin W -graph representations do provide suitable models for the “cell representations”. This fact raises a general question about W -graph representations which is formulated as Conjecture 4.5.

2. Cellular bases and cell representations

Let W be an irreducible finite Weyl group with generating set S . Let $R \subseteq \mathbb{C}$ be a subring and $A = R[v, v^{-1}]$ the ring of Laurent polynomials in an indeterminate v . Let \mathcal{H} be the corresponding 1-parameter Iwahori–Hecke algebra over A . As an A -module, \mathcal{H} is free with basis $\{T_w \mid w \in W\}$; the multiplication is given by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w) + 1, \\ uT_{sw} + (u-1)T_w & \text{if } l(sw) = l(w) - 1, \end{cases}$$

where $u = v^2$, $s \in S$ and $w \in W$. Here, $l(w)$ denotes the length of $w \in W$. For the general theory of Iwahori–Hecke algebras, we refer to [20]. These algebras, and their specialisations, play an important role in the representation theory of finite reductive groups; see, for example, [32, Chap. 0], [14].

In order to specify a *cell datum* for \mathcal{H} in the sense of Graham and Lehrer [23, Def. 1.1], we must specify a quadruple $(\Lambda, M, C, *)$ satisfying the following conditions.

(C1) Λ is a partially ordered set, $\{M(\lambda) \mid \lambda \in \Lambda\}$ is a collection of finite sets and

$$C: \prod_{\lambda \in \Lambda} M(\lambda) \times M(\lambda) \rightarrow \mathcal{H}$$

is an injective map whose image is an A -basis of \mathcal{H} ;

(C2) If $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$, write $C(\mathfrak{s}, \mathfrak{t}) = C_{\mathfrak{s}, \mathfrak{t}}^\lambda \in \mathcal{H}$. Then $*$: $\mathcal{H} \rightarrow \mathcal{H}$ is an A -linear anti-involution such that $(C_{\mathfrak{s}, \mathfrak{t}}^\lambda)^* = C_{\mathfrak{t}, \mathfrak{s}}^\lambda$.

(C3) If $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$, then for any element $h \in \mathcal{H}$ we have

$$hC_{\mathfrak{s}, \mathfrak{t}}^\lambda \equiv \sum_{\mathfrak{s}' \in M(\lambda)} r_h(\mathfrak{s}', \mathfrak{s}) C_{\mathfrak{s}', \mathfrak{t}}^\lambda \pmod{\mathcal{H}(< \lambda)},$$

where $r_h(\mathfrak{s}', \mathfrak{s}) \in A$ is independent of \mathfrak{t} and where $\mathcal{H}(< \lambda)$ is the A -submodule of \mathcal{H} generated by $\{C_{\mathfrak{s}'', \mathfrak{t}''}^\mu \mid \mu < \lambda; \mathfrak{s}'', \mathfrak{t}'' \in M(\mu)\}$.

For this purpose, we first need to recall some basic facts about the representations of W and $\mathcal{H}_K = K \otimes_A \mathcal{H}$, where K is the field of fractions of A .

It is known that \mathbb{Q} is a splitting field for W ; see, for example, [20, 6.3.8]. We will write

$$\text{Irr}(W) = \{E^\lambda \mid \lambda \in \Lambda\}, \quad d_\lambda = \dim E^\lambda,$$

for the set of irreducible representations of W (up to equivalence), where Λ is some finite indexing set. Now, the algebra \mathcal{H}_K is known to be split semisimple; see [20, 9.3.5]. Furthermore, by Tits' Deformation Theorem, the irreducible representations of \mathcal{H}_K (up to isomorphism) are in bijection with the irreducible representations of W ; see [20, 8.1.7]. Thus, we can write

$$\text{Irr}(\mathcal{H}_K) = \{E_v^\lambda \mid \lambda \in \Lambda\}.$$

The correspondence $E^\lambda \leftrightarrow E_v^\lambda$ is uniquely determined by the following condition:

$$\text{trace}(w, E^\lambda) = \text{trace}(T_w, E_v^\lambda) \Big|_{v=1} \quad \text{for all } w \in W;$$

note that $\text{trace}(T_w, E_v^\lambda) \in A$ for all $w \in W$.

The algebra \mathcal{H} is *symmetric* with respect to the trace form $\tau: \mathcal{H} \rightarrow A$ defined by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $1 \neq w \in W$. Hence we have the following orthogonality relations for the irreducible representations of \mathcal{H}_K :

$$\sum_{w \in W} u^{-l(w)} \text{trace}(T_w, E_v^\lambda) \text{trace}(T_{w^{-1}}, E_v^\mu) = \begin{cases} d_\lambda \mathbf{c}_\lambda & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu, \end{cases}$$

where $0 \neq \mathbf{c}_\lambda \in \mathbb{Z}[u, u^{-1}]$; see [20, 8.1.7 and 9.3.6]. Following Lusztig, we write

$$\mathbf{c}_\lambda = f_\lambda u^{-\mathbf{a}_\lambda} + \text{combination of strictly higher powers of } u,$$

where $\mathbf{a}_\lambda, f_\lambda$ are integers such that $\mathbf{a}_\lambda \geq 0$ and $f_\lambda > 0$; see [20, 9.4.7]. These integers are explicitly known for all types of W ; see Lusztig [31, Chap. 4] or [32, Chap. 22].

Remark 2.1. Since we are in the equal parameter case, the Laurent polynomials \mathbf{c}_λ have the following properties: Each \mathbf{c}_λ divides the Poincaré polynomial $P_W = \sum_{w \in W} u^{l(w)}$ in $\mathbb{Q}[u, u^{-1}]$; furthermore, we have

$$\mathbf{c}_\lambda = f_\lambda u^{-\mathbf{a}_\lambda} \tilde{\mathbf{c}}_\lambda \quad \text{where } \tilde{\mathbf{c}}_\lambda \in \mathbb{Z}[u] \text{ is monic and divides } P_W.$$

(For these facts, see [20, 9.3.6] and the references there.) It is well-known (see, for example, [3, §9.4]) that

$$P_W = \prod_{1 \leq i \leq |S|} \frac{u^{d_i} - 1}{u - 1}$$

where $d_1, \dots, d_{|S|}$ are the so-called *degrees* of W ; we have $|W| = d_1 \cdots d_{|S|}$. By [3, §10.2], the degrees for the various types of W are given as follows:

Type	degrees d_i	Type	degrees d_i
A_{n-1}	$2, 3, 4, \dots, n$	G_2	$2, 6$
B_n, C_n	$2, 4, 6, \dots, 2n$	F_4	$2, 6, 8, 12$
D_n	$2, 4, 6, \dots, 2(n-1), n$	E_6	$2, 5, 6, 8, 9, 12$
		E_7	$2, 6, 8, 10, 12, 14, 18$
		E_8	$2, 8, 12, 14, 18, 20, 24, 30$

We are now ready to define a “cell datum” of \mathcal{H} . The required quadruple $(\Lambda, M, C, *)$ is given as follows. Let Λ be an indexing set for the irreducible representations of W , as above. For $\lambda \in \Lambda$, we set $M(\lambda) = \{1, \dots, d_\lambda\}$. Using the \mathbf{a} -invariants, we define a partial order \preceq on Λ by

$$\lambda \preceq \mu \quad \stackrel{\text{def}}{\iff} \quad \lambda = \mu \quad \text{or} \quad \mathbf{a}_\lambda > \mathbf{a}_\mu.$$

Thus, Λ is ordered according to *decreasing* \mathbf{a} -value. Next, we define an A -linear anti-involution $*$: $\mathcal{H} \rightarrow \mathcal{H}$ by $T_w^* = T_{w^{-1}}$ for all $w \in W$. Thus, $T_w^* = T_w^\flat$ in the notation of [32, 3.4].

The trickiest part is, of course, the definition of the basis elements $C_{\mathfrak{s}, \mathfrak{t}}^\lambda$ for $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$. Let $\{c_w \mid w \in W\}$ be the Kazdan–Lusztig basis of \mathcal{H} , as constructed in [32, Theorem 5.2]. Given $x, y \in W$, we write $c_x c_y = \sum_{z \in W} h_{x,y,z} c_z$ where $h_{x,y,z} \in A$. Following Lusztig [32, 13.6], we use the structure constants $h_{x,y,z}$ to define a function $\mathbf{a}: W \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\mathbf{a}(z) := \min\{i \geq 0 \mid v^i h_{x,y,z} \in \mathbb{Z}[v] \text{ for all } x, y \in W\} \quad \text{for all } z \in W.$$

As in [*loc. cit.*], we usually work with the elements c_w^\dagger obtained by applying the unique A -algebra involution $\mathcal{H} \rightarrow \mathcal{H}$, $h \mapsto h^\dagger$ such that $T_s^\dagger = -T_s^{-1}$ for any $s \in S$; see [32, 3.5]. We can now state:

Theorem 2.2 (Geck [16, Theorem 3.1]). *Assume that the subring $R \subseteq \mathbb{C}$ is chosen such that all bad primes for W are invertible in R . Then there is a cell datum $(\Lambda, M, C, *)$ for \mathcal{H} where $\Lambda, M, *$ are as specified above and, for each $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$, the element $C_{\mathfrak{s}, \mathfrak{t}}^\lambda$ is a \mathbb{Z} -linear combination of basis elements c_w^\dagger where $\mathbf{a}(w) = \mathbf{a}_\lambda$.*

Here, a prime number p is called *bad* for W if p divides f_λ for some $\lambda \in \Lambda$. Otherwise, p is called *good*. This corresponds to the familiar definition of “bad” primes; see Lusztig [31, Chap. 4]. The conditions for being good for the various types of W are as follows:

$$\begin{aligned} A_n &: \text{no condition,} \\ B_n, C_n, D_n &: p \neq 2, \\ G_2, F_4, E_6, E_7 &: p \neq 2, 3, \\ E_8 &: p \neq 2, 3, 5. \end{aligned}$$

For the rest of this paper, we shall now make the definite choice where the ring R consists of all fractions $a/b \in \mathbb{Q}$ such that $a \in \mathbb{Z}$ and $0 \neq b \in \mathbb{Z}$ is divisible by bad primes only.

Remark 2.3. For future reference, we remark that, if $h \in \mathcal{H}$ is a $\mathbb{Z}[v, v^{-1}]$ -linear combination of basis elements $\{T_w \mid w \in W\}$, then we also have

$$r_h(\mathfrak{s}', \mathfrak{s}) \in \mathbb{Z}[v, v^{-1}] \quad \text{for all } \lambda \in \Lambda \text{ and } \mathfrak{s}, \mathfrak{s}' \in M(\lambda);$$

see the explicit formula for $r_h(\mathfrak{s}', \mathfrak{s})$ in Step 3 of the proof of [16, Theorem 3.1].

Following Graham and Lehrer [23], we can perform the following constructions. Given $\lambda \in \Lambda$, let W^λ be a free A -module with basis $\{C_\mathfrak{s} \mid \mathfrak{s} \in M(\lambda)\}$. Then W^λ is a left \mathcal{H} -module, where the action is given by

$$h.C_\mathfrak{s} = \sum_{\mathfrak{s}' \in M(\lambda)} r_h(\mathfrak{s}', \mathfrak{s}) C_{\mathfrak{s}'}.$$

Furthermore, we can define a symmetric bilinear form $\phi^\lambda: W^\lambda \times W^\lambda \rightarrow A$ by

$$\phi^\lambda(C_{\mathfrak{s}}, C_{\mathfrak{t}}) = r_h(\mathfrak{s}, \mathfrak{t}) \quad \text{where } \mathfrak{s}, \mathfrak{t} \in M(\lambda) \text{ and } h = C_{\mathfrak{s}, \mathfrak{t}}^\lambda.$$

We have $\phi^\lambda(T_w.C_{\mathfrak{s}}, C_{\mathfrak{t}}) = \phi^\lambda(C_{\mathfrak{s}}, T_{w^{-1}}.C_{\mathfrak{t}})$ for all $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$ and $w \in W$; see [23, Prop. 2.4].

The modules $\{W^\lambda \mid \lambda \in \Lambda\}$ are called the *cell representations*, or *cell modules*, of \mathcal{H} . Extending scalars from A to K , we obtain modules $W_K^\lambda = K \otimes_A W^\lambda$ for \mathcal{H}_K . By the discussion in [16, Exp. 4.4], we have

$$\text{Irr}(\mathcal{H}_K) = \{W_K^\lambda \mid \lambda \in \Lambda\} \quad \text{and} \quad W_K^\lambda \cong E_v^\lambda \quad \text{for all } \lambda \in \Lambda.$$

Now let $\theta: A \rightarrow k$ be a ring homomorphism into a field k ; note that the characteristic of k will be either 0 or a prime p which is not bad for W . By extension of scalars, we obtain a k -algebra $\mathcal{H}_k(W, \xi) = k \otimes_A \mathcal{H}$ where $\xi := \theta(u) \in k$. Explicitly, $\mathcal{H}_k(W, \xi)$ has a basis $\{T_w \mid w \in W\}$ and the multiplication is given by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w) + 1, \\ \xi T_{sw} + (\xi - 1)T_w & \text{if } l(sw) = l(w) - 1, \end{cases}$$

where $s \in S$ and $w \in W$. The algebra $\mathcal{H}_k(W, \xi)$ is called a *specialisation* of \mathcal{H} . Let $\text{Irr}(\mathcal{H}_k(W, \xi))$ be the set of irreducible representations of $\mathcal{H}_k(W, \xi)$, up to isomorphism.

Now, we also obtain cell modules $W_\xi^\lambda = k \otimes_A W^\lambda$ ($\lambda \in \Lambda$) for $\mathcal{H}_k(W, \xi)$, which may no longer be irreducible. Denoting by ϕ_ξ^λ the induced bilinear form on W_ξ^λ , we set

$$L_\xi^\lambda = W_\xi^\lambda / \text{rad}(\phi_\xi^\lambda).$$

Then, by the general theory of cellular algebras in [23, §3], each L_ξ^λ is either $\{0\}$ or an absolutely simple $\mathcal{H}_k(W, \xi)$ -module, and we have

$$\text{Irr}(\mathcal{H}_k(W, \xi)) = \{L_\xi^\mu \mid \mu \in \Lambda_\xi^\circ\} \quad \text{where} \quad \Lambda_\xi^\circ := \{\lambda \in \Lambda \mid L_\xi^\lambda \neq 0\}.$$

In particular, this shows that the algebra $\mathcal{H}_k(W, \xi)$ is split. Furthermore, denoting by $(W_\xi^\lambda : L_\xi^\mu)$ the multiplicity of L_ξ^μ as a composition factor of W_ξ^λ , we have

$$(\Delta) \quad \begin{cases} (W_\xi^\mu : L_\xi^\mu) = 1 & \text{for any } \mu \in \Lambda_\xi^\circ, \\ (W_\xi^\lambda : L_\xi^\mu) = 0 & \text{unless } \lambda = \mu \text{ or } \mathbf{a}_\mu < \mathbf{a}_\lambda. \end{cases}$$

Thus, the theory of cellular algebras provides a general method for constructing the irreducible representations of the specialized algebra $\mathcal{H}_k(W, \xi)$.

Proposition 2.4. *Assume that $P_W(\xi) \neq 0$. Then $\mathcal{H}_k(W, \xi)$ is semisimple, $\Lambda = \Lambda_\xi^\circ$ and $W_\xi^\lambda = L_\xi^\lambda$ for all $\lambda \in \Lambda$.*

Proof. Recall from Remark 2.1 that, for each $\lambda \in \Lambda$, we have $\mathbf{c}_\lambda = f_\lambda u^{-\mathbf{a}_\lambda} \tilde{\mathbf{c}}_\lambda$ where $\tilde{\mathbf{c}}_\lambda \in \mathbb{Z}[u]$ is monic and divides P_W . Hence, since the characteristic of k is either 0 or a good prime for W , our assumption $P_W(\xi) \neq 0$ implies that we also have $\theta(\mathbf{c}_\lambda) \neq 0$ for all $\lambda \in \Lambda$. A general semisimplicity criterion for symmetric algebras (see [20, 7.4.7]) then shows that $\mathcal{H}_k(W, \xi)$ is semisimple, a result first proved by Gyoja–Uno [25]. The remaining statements concerning the cell representations are contained in [23, 3.8]. \square

Corollary 2.5. *Let $\lambda \in \Lambda$ and G^λ be the Gram matrix of the invariant bilinear form ϕ^λ with respect to the standard basis of W^λ . Then $0 \neq \det(G^\lambda) \in \mathbb{Z}[v, v^{-1}]$. Furthermore, let $0 \neq q \in \mathbb{Z}[v, v^{-1}]$ be irreducible such that q divides $\det(G^\lambda)$. Then either $\pm q$ is a bad prime number or q divides P_W .*

Proof. First note that, by Remark 2.3, all entries of G^λ lie in $\mathbb{Z}[v, v^{-1}]$. Furthermore, by Proposition 2.4, we have $\det(G^\lambda) \neq 0$. Now consider the prime ideal (q) and let F be the field of fractions of $A/(q)$. Then we have a specialisation $\alpha: A \rightarrow F$. Let $\mathcal{H}_F(W, \alpha(u))$ be the specialised algebra. Let G_F^λ be the matrix obtained by applying α to all coefficients of G^λ . Then G_F^λ is the Gram matrix of the induced bilinear form ϕ_F^λ on the specialised cell module W_F^λ . If q divides $\det(G^\lambda)$, then $\det(G_F^\lambda) = 0$ and so $\mathcal{H}_F(W, \alpha(u))$ will not be semisimple; see [23, 3.8]. By the general semisimplicity criterion in [20, 7.4.7], we deduce that $\alpha(\mathbf{c}_\mu) = 0$ for some $\mu \in \Lambda$. Now there are two cases.

If $q \in \mathbb{Z}$, then this implies that q must divide f_μ and so $\pm q$ is a bad prime.

If q is an irreducible non-constant polynomial, then q must divide \mathbf{c}_μ . By Remark 2.1, \mathbf{c}_μ divides P_W . Hence, we deduce that q divides P_W . \square

Example 2.6. Let W be of type A_{n-1} . Then W can be identified with the symmetric group \mathfrak{S}_n and Λ consists of all partitions $\lambda \vdash n$. A special feature of this case is that $f_\lambda = 1$ for all $\lambda \in \Lambda$. By [16, Exp. 4.2], the linear combinations in Theorem 2.2 will only have one non-zero term, with coefficient 1, i.e., the Kazhdan–Lusztig basis itself is a cellular basis. More precisely, for $\lambda \in \Lambda$, let w_λ be the longest element in the corresponding Young subgroup \mathfrak{S}_λ of $W = \mathfrak{S}_n$. Now, by [29, §5], the Kazhdan–Lusztig left and right cells of W are given by the Robinson–Schensted correspondence. This explicit description shows that, if \mathfrak{C}_λ denotes the left cell containing w_λ , we have

$$\mathfrak{C}_\lambda = \{d(\mathfrak{s})w_\lambda \mid \mathfrak{s} \in M(\lambda)\}$$

where the elements $d(\mathfrak{s})$ ($\mathfrak{s} \in M(\lambda)$) are certain distinguished left coset representatives of \mathfrak{S}_λ in $W = \mathfrak{S}_n$. Furthermore, given $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$, there is a unique $w_\lambda(\mathfrak{s}, \mathfrak{t}) \in W$ such that $w_\lambda(\mathfrak{s}, \mathfrak{t})$ lies in the same right cell as $d(\mathfrak{s})w_\lambda$ and in the same left cell as $w_\lambda d(\mathfrak{t})^{-1}$. (See also [15, Rem. 3.9, Cor. 5.6] for further details.) With this notation, [16, Exp. 4.2] shows that

$$C_{\mathfrak{s}, \mathfrak{t}}^\lambda = c_{w_\lambda(\mathfrak{s}, \mathfrak{t})}^\dagger \quad \text{for all } \lambda \vdash n \text{ and } \mathfrak{s}, \mathfrak{t} \in M(\lambda).$$

McDonough and Pallikaros [34] showed that the cell modules W^λ are naturally isomorphic to the Dipper–James Specht modules. The invariant bilinear form on W^λ is given by

$$\phi^\lambda(C_{\mathfrak{s}}, C_{\mathfrak{t}}) = h_{w_\lambda d(\mathfrak{s})^{-1}, d(\mathfrak{t})w_\lambda, w_\lambda} \quad \text{for all } \mathfrak{s}, \mathfrak{t} \in M(\lambda).$$

For connections of these bilinear forms with the topology of Springer fibres, see Fung [7].

Thus, for general \mathcal{H} , the cell modules W^λ arising from Theorem 2.2 can indeed be regarded as analogues of the Dipper–James Specht modules in type A_{n-1} .

Example 2.7. Let W be the Weyl group of type G_2 where $S = \{s_1, s_2\}$ and $(s_1 s_2)^6 = 1$. We have $\text{Irr}(W) = \{\mathbf{1}, \varepsilon_1, \varepsilon_2, \varepsilon, r, r'\}$ where $\mathbf{1}$ is the unit representation, ε is the sign representation, $\varepsilon_1, \varepsilon_2$ have dimension one, r is the reflection representation and r' is another representation of dimension two. The invariants \mathbf{a}_λ and f_λ are given by

$$\begin{aligned} \mathbf{a}_\mathbf{1} &= 0, & \mathbf{a}_{\varepsilon_1} &= \mathbf{a}_{\varepsilon_2} = \mathbf{a}_r = \mathbf{a}_{r'} = 1, & \mathbf{a}_\varepsilon &= 6; \\ f_\mathbf{1} &= f_\varepsilon = 1, & f_{\varepsilon_1} &= f_{\varepsilon_2} = 3, & f_r &= 6, & f_{r'} &= 2. \end{aligned}$$

Hence, the bad primes are 2 and 3. A cellular basis as in Theorem 2.2 is given as follows:

$$\begin{aligned}
C_{1,1}^1 &= c_1^\dagger, & C_{1,1}^\varepsilon &= c_{w_0}^\dagger, \\
C_{1,1}^{\varepsilon_1} &= c_{s_2}^\dagger - c_{s_2 s_1 s_2}^\dagger + c_{s_2 s_1 s_2 s_1 s_2}^\dagger, & C_{1,1}^{\varepsilon_2} &= c_{s_1}^\dagger - c_{s_1 s_2 s_1}^\dagger + c_{s_1 s_2 s_1 s_2 s_1}^\dagger, \\
C_{1,1}^r &= 3c_{s_1}^\dagger + 6c_{s_1 s_2 s_1}^\dagger + 3c_{s_1 s_2 s_1 s_2 s_1}^\dagger, & C_{1,1}^{r'} &= c_{s_1}^\dagger - c_{s_1 s_2 s_1 s_2 s_1}^\dagger, \\
C_{1,2}^r &= -3c_{s_1 s_2}^\dagger - 3c_{s_1 s_2 s_1 s_2}^\dagger, & C_{1,2}^{r'} &= -c_{s_1 s_2}^\dagger + c_{s_1 s_2 s_1 s_2}^\dagger, \\
C_{2,1}^r &= -3c_{s_2 s_1}^\dagger - 3c_{s_2 s_1 s_2 s_1}^\dagger, & C_{2,1}^{r'} &= -c_{s_2 s_1}^\dagger + c_{s_2 s_1 s_2 s_1}^\dagger, \\
C_{2,2}^r &= c_{s_2}^\dagger + 2c_{s_2 s_1 s_2}^\dagger + c_{s_2 s_1 s_2 s_1 s_2}^\dagger, & C_{2,2}^{r'} &= c_{s_2}^\dagger - c_{s_2 s_1 s_2 s_1 s_2}^\dagger.
\end{aligned}$$

To find these expressions, we perform computations similar to those in [16, Exp. 4.3] (where type B_2 was considered). Once this is done, one can then also check directly that the above elements form a cellular basis. The Gram matrices of the invariant bilinear forms on the cell representations W^λ are given by

$$\begin{aligned}
G^1 &= \begin{bmatrix} 1 \end{bmatrix}, & G^\varepsilon &= \begin{bmatrix} v^{-6} P_W \end{bmatrix}, & G^{\varepsilon_1} &= G^{\varepsilon_2} = \begin{bmatrix} 3(v + v^{-1}) \end{bmatrix}, \\
G^r &= \begin{bmatrix} 18(v + v^{-1}) & -18 \\ -18 & 6(v + v^{-1}) \end{bmatrix}, & G^{r'} &= \begin{bmatrix} 2(v + v^{-1}) & -2 \\ -2 & 2(v + v^{-1}) \end{bmatrix},
\end{aligned}$$

where $P_W = (v^{12} - 1)(v^4 - 1)/(v^2 - 1)^2$ is the Poincaré polynomial of W .

Now let $\theta: A \rightarrow k$ be a specialisation; note that the characteristic of k will be either 0 or a prime $\neq 2, 3$. Let $e \geq 2$ be minimal such that $1 + \xi + \xi^2 + \dots + \xi^{e-1} = 0$. Thus, either $\xi = 1$ and e is the characteristic of k , or e is the multiplicative order of ξ in k^\times . We see that the above Gram matrices remain non-singular after specialisation unless $\xi \neq 1$ and $e \in \{2, 3, 6\}$. Thus, we obtain non-trivial decomposition numbers only for $e \in \{2, 3, 6\}$. In these cases, the sets Λ_ξ° and the dimensions of L_ξ^μ for $\mu \in \Lambda_\xi^\circ$ are given as follows.

$e = 2$			$e = 3$			$e = 6$		
Λ_ξ°	\mathbf{a}_μ	$\dim L_\xi^\mu$	Λ_ξ°	\mathbf{a}_μ	$\dim L_\xi^\mu$	Λ_ξ°	\mathbf{a}_μ	$\dim L_\xi^\mu$
$\mathbf{1}$	0	1	$\mathbf{1}$	0	1	$\mathbf{1}$	0	1
r	1	2	ε_1	1	1	ε_1	1	1
r'	1	2	ε_2	1	1	ε_2	1	1
			r	1	2	r	1	1
			r'	1	1	r'	1	2

In particular, we notice that the classification of the irreducible representations and their dimensions only depend on e , but not on the particular value of ξ or the characteristic of k . Thus, we have verified in a particular example the general phenomenon which is expressed in James' conjecture.

Remark 2.8. The decomposition matrix D_ξ can also be interpreted in the framework of Brauer's modular representation theory of associative algebras; see [6, §I.1.17]. Indeed, let us assume that k is the field of fractions of the image of θ . By [20, Exc. 7.8], there exists a discrete valuation ring $\mathcal{O} \subseteq K$ with maximal ideal \mathfrak{p} such that $A \subseteq \mathcal{O}$ and $\mathfrak{p} \cap A = \ker(\theta)$. Let $k_{\mathfrak{p}} \supseteq k$ be the residue field of \mathcal{O} . Since $\mathcal{H}_k(W, \xi)$ is split, the scalar extension from k

to $k_{\mathfrak{p}}$ induces a bijection $\text{Irr}(\mathcal{H}_k(W, \xi)) \xrightarrow{\sim} \text{Irr}(\mathcal{H}_{k_{\mathfrak{p}}}(W, \xi))$. Identifying $\text{Irr}(\mathcal{H}_k(W, \xi))$ and $\text{Irr}(\mathcal{H}_{k_{\mathfrak{p}}}(W, \xi))$ via this isomorphism, we obtain a well-defined decomposition map

$$d_{\xi}: R_0(\mathcal{H}_K) \rightarrow R_0(\mathcal{H}_k(W, \xi))$$

where $R_0(\mathcal{H}_K)$ and $R_0(\mathcal{H}_k(W, \xi))$ denote the Grothendieck groups of finite-dimensional representations of \mathcal{H}_K and $\mathcal{H}_k(W, \xi)$, respectively. Since each cell representation W^{λ} is defined over A and $W_K^{\lambda} \cong E_v^{\lambda}$, we conclude that

$$d_{\xi}([E_v^{\lambda}]) = \sum_{\mu \in \Lambda_{\xi}^{\circ}} (W_{\xi}^{\lambda} : L_{\xi}^{\mu}) [L_{\xi}^{\mu}] \quad \text{for all } \lambda \in \Lambda,$$

where $[E_v^{\lambda}]$, $[L_{\xi}^{\mu}]$ denote the classes of E_v^{λ} , L_{ξ}^{μ} in the respective Grothendieck groups. (Note that, by [4, Ex. 6.16], we do not need to pass to the completion of \mathcal{O} , as is usually done in Brauer's modular representation theory.)

Definition 2.9. The *Brauer graph* of \mathcal{H} with respect to $\theta: A \rightarrow k$ is the graph with vertices labelled by the elements of Λ and edges given as follows. Let $\lambda \neq \lambda'$ in Λ . Then the vertices labelled by λ and λ' are joined by an edge if there exists some $\mu \in \Lambda_{\xi}^{\circ}$ such that $(W_{\xi}^{\lambda} : L^{\mu}) \neq 0$ and $(W_{\xi}^{\lambda'} : L^{\mu}) \neq 0$. The connected components of this graph define a partition of Λ which are called the ξ -blocks of Λ (or of $\text{Irr}(\mathcal{H}_K)$ or of $\text{Irr}(W)$).

Let $\Lambda = \Lambda_1 \amalg \Lambda_2 \amalg \cdots \amalg \Lambda_r$ be the partition of Λ into ξ -blocks. Then we also have

$$\Lambda_{\xi}^{\circ} = \Lambda_{\xi,1}^{\circ} \amalg \Lambda_{\xi,2}^{\circ} \amalg \cdots \amalg \Lambda_{\xi,r}^{\circ} \quad \text{where} \quad \Lambda_{\xi,i}^{\circ} := \Lambda_i \cap \Lambda_{\xi}^{\circ}.$$

If we order the elements of Λ and of Λ_{ξ}° accordingly, we obtain a block diagonal shape for D_{ξ} :

$$D_{\xi} = \begin{pmatrix} D_{\xi,1} & 0 & \cdots & 0 \\ 0 & D_{\xi,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_{\xi,r} \end{pmatrix},$$

where $D_{\xi,i}$ has rows and columns labelled by the elements of Λ_i and $\Lambda_{\xi,i}^{\circ}$, respectively. Thus, in order to describe the set Λ_{ξ}° and the matrix D_{ξ} , we can proceed block by block. Note that, by Remark 2.8, the blocks of \mathcal{H} as defined above really correspond to blocks in the sense of Brauer's modular representation theory.

3. The general version of James' conjecture

We keep the general setting of the previous section. Let \mathcal{H} be an Iwahori–Hecke algebra associated with a finite Weyl group W , defined over the ring $A = R[v, v^{-1}]$ where $R \subseteq \mathbb{Q}$ is fixed as in the remarks just after Theorem 2.2. Then we have a cellular basis $\{C_{s,t}^{\lambda}\}$ and cell representations $\{W^{\lambda} \mid \lambda \in \Lambda\}$ for \mathcal{H} .

Now let $\theta: A \rightarrow k$ be a ring homomorphism into a field k . Note that the characteristic of k will be either 0 or a prime p which is not bad for W . We obtain a corresponding specialised algebra $\mathcal{H}_k(W, \xi)$ where $\xi = \theta(u) \in k^{\times}$. Recall that

$$\text{Irr}(\mathcal{H}_k(W, \xi)) = \{L_{\xi}^{\mu} \mid \mu \in \Lambda_{\xi}^{\circ}\}.$$

As in Remark 2.8, we have a decomposition map $d_\xi: R_0(\mathcal{H}_K) \rightarrow R_0(\mathcal{H}_k(W, \xi))$ such that

$$d_\xi([E_v^\lambda]) = \sum_{\mu \in \Lambda_\xi^\circ} (W_\xi^\lambda : L_\xi^\mu) [L_\xi^\mu] \quad \text{for all } \lambda \in \Lambda.$$

Following Dipper–James [5], we set

$$e = \min\{i \geq 2 \mid 1 + \xi + \xi^2 + \cdots + \xi^{i-1} = 0\}.$$

(We set $e = \infty$ if no such i exists.) We assume from now on that $\text{char}(k) = \ell > 0$ and $e < \infty$. Let $\zeta_e = \sqrt[e]{1} \in \mathbb{C}$ and consider the Iwahori–Hecke algebra $\mathcal{H}_{\mathbb{C}}(W, \zeta_e)$ arising from the specialisation

$$\theta_e: A \rightarrow \mathbb{C}, \quad v \mapsto \zeta_{2e} = \sqrt[e]{1}.$$

We can apply the previous discussion to the algebra $\mathcal{H}_{\mathbb{C}}(W, \zeta_e)$ as well. Thus, we have

$$\text{Irr}(\mathcal{H}_{\mathbb{C}}(W, \zeta_e)) = \{L_{\zeta_e}^\mu \mid \mu \in \Lambda_{\zeta_e}^\circ\}.$$

Furthermore, there is a decomposition map $d_{\zeta_e}: R_0(\mathcal{H}_K) \rightarrow R_0(\mathcal{H}_{\mathbb{C}}(W, \zeta_e))$ such that

$$d_{\zeta_e}([E_v^\lambda]) = \sum_{\mu \in \Lambda_{\zeta_e}^\circ} (W_{\zeta_e}^\lambda : L_{\zeta_e}^\mu) [L_{\zeta_e}^\mu] \quad \text{for all } \lambda \in \Lambda.$$

We will want to compare the representations of $\mathcal{H}_k(W, \xi)$ and $\mathcal{H}_{\mathbb{C}}(W, \zeta_e)$. For this purpose, the following remark will be relevant.

Remark 3.1. For any $d \geq 1$, we denote by $\Phi_d \in \mathbb{Z}[u]$ the d th cyclotomic polynomial. Note that we have

$$\Phi_d(v^2) = \begin{cases} \Phi_{2d}(v) & \text{if } d \text{ is even,} \\ \Phi_d(v)\Phi_d(-v) & \text{if } d \text{ is odd.} \end{cases}$$

Now, in view of the definition of e , it is clear that $\Phi_e(\xi) = 0$. Furthermore, note that $\theta(v)^2 = \xi$. Hence, choosing a square root of ξ in k^\times appropriately, we can assume that $\Phi_{2e}(\theta(v)) = 0$. (If $\text{char}(k) \neq 2$, we also have $\Phi_e(\theta(v)) \neq 0$.) Consequently, there exists a ring homomorphism $R[\zeta_{2e}] \rightarrow k$, $r \mapsto \bar{r}$, such that $\theta(a) = \overline{\theta_e(a)}$ for all $a \in A$. Let $\mathcal{O} \subseteq \mathbb{Q}(\zeta_{2e})$ be the localisation of $R[\zeta_{2e}]$ in the prime ideal $\mathfrak{q} = \{r \in R[\zeta_{2e}] \mid \bar{r} = 0\}$. Then \mathcal{O} is a discrete valuation ring whose residue field can be identified with a subfield of k . By “ \mathfrak{q} -modular reduction” (see [6, §I.1.17]), we obtain a well-defined decomposition map

$$d_\xi^e: R_0(\mathcal{H}_{\mathbb{Q}(\zeta_{2e})}(W, \zeta_e)) \rightarrow R_0(\mathcal{H}_k(W, \xi)).$$

Note that the scalar extension from $\mathbb{Q}(\zeta_{2e})$ to \mathbb{C} defines a bijection

$$\text{Irr}(\mathcal{H}_{\mathbb{Q}(\zeta_{2e})}(W, \zeta_e)) \xrightarrow{\sim} \text{Irr}(\mathcal{H}_{\mathbb{C}}(W, \zeta_e)).$$

Via this bijection, we can identify $R_0(\mathcal{H}_{\mathbb{Q}(\zeta_{2e})}(W, \zeta_e))$ and $R_0(\mathcal{H}_{\mathbb{C}}(W, \zeta_e))$, and regard d_ξ^e as a map from $R_0(\mathcal{H}_{\mathbb{C}}(W, \zeta_e))$ to $R_0(\mathcal{H}_k(W, \xi))$. Let us write

$$d_\xi^e([L_{\zeta_e}^\nu]) = \sum_{\mu \in \Lambda_\xi^\circ} a_{\nu\mu} [L_\xi^\mu] \quad \text{for any } \nu \in \Lambda_{\zeta_e}^\circ,$$

where $a_{\nu\mu} \in \mathbb{Z}_{\geq 0}$. Following James [28], the matrix $A_\xi^e := (a_{\nu\mu})$ is called the *adjustment matrix* associated to the specialisation θ . By a general factorisation result for decomposition maps, we have $d_\xi = d_\xi^e \circ d_{\zeta_e}$ or, in other words,

$$(W_\xi^\lambda : L_\xi^\mu) = \sum_{\nu \in \Lambda_{\zeta_e}^\circ} a_{\nu\mu} (W_{\zeta_e}^\lambda : L_{\zeta_e}^\nu) \quad \text{for all } \lambda \in \Lambda \text{ and } \mu \in \Lambda_\xi^\circ.$$

This result first appeared in [8, Theorem 5.3]; see also [21, Prop. 2.5], [11, Prop. 2.6] for analogous statements in more general situations.

Lemma 3.2. *In the above setting, the following hold.*

- (a) *Given $\mu \in \Lambda_\xi^\circ$ and $\nu \in \Lambda_{\zeta_e}^\circ$, we have $a_{\nu\mu} = 0$ unless $\nu = \mu$ or $\mathbf{a}_\mu < \mathbf{a}_\nu$.*
- (b) *We have $\Lambda_\xi^\circ \subseteq \Lambda_{\zeta_e}^\circ$ and $a_{\mu\mu} = 1$ for all $\mu \in \Lambda_\xi^\circ$. In particular, we have $\Lambda_\xi^\circ = \Lambda_{\zeta_e}^\circ$ if these two sets have the same cardinality.*
- (c) *We have $\dim L_\xi^\mu \leq \dim L_{\zeta_e}^\mu$ for all $\mu \in \Lambda_\xi^\circ$.*

Proof. Let $\lambda \in \Lambda$, $\mu \in \Lambda_\xi^\circ$ and $\nu \in \Lambda_{\zeta_e}^\circ$. Recall the relations (Δ) from Section 2: if $(W_\xi^\lambda : L_\xi^\mu) \neq 0$, then $\mathbf{a}_\mu \leq \mathbf{a}_\lambda$ with equality only for $\lambda = \mu$; furthermore, $(W_\xi^\mu : L_\xi^\mu) = 1$. A similar statement holds for the decomposition numbers $(W_{\zeta_e}^\lambda : L_{\zeta_e}^\nu)$.

(a) Assume that $a_{\nu\mu} \neq 0$. Then, since $(W_{\zeta_e}^\nu : L_{\zeta_e}^\nu) = 1$, we have

$$(W_\xi^\nu : L_\xi^\mu) = \sum_{\nu' \in \Lambda_{\zeta_e}^\circ} a_{\nu'\mu} (W_{\zeta_e}^\nu : L_{\zeta_e}^{\nu'}) > 0$$

and so the relations (Δ) imply that $\nu = \mu$ or $\mathbf{a}_\mu < \mathbf{a}_\nu$.

(b) We have $1 = (W_\xi^\mu : L_\xi^\mu) = \sum_{\nu' \in \Lambda_{\zeta_e}^\circ} a_{\nu'\mu} (W_{\zeta_e}^\mu : L_{\zeta_e}^{\nu'})$. So there exists some $\nu' \in \Lambda_{\zeta_e}^\circ$ such that $a_{\nu'\mu} \neq 0$ and $(W_{\zeta_e}^\mu : L_{\zeta_e}^{\nu'}) \neq 0$. Consequently, using (a) and the relations (Δ) , we have $\mathbf{a}_\mu \leq \mathbf{a}_{\nu'} \leq \mathbf{a}_\mu$ and so $\mathbf{a}_\mu = \mathbf{a}_{\nu'}$. Thus, we must have $\mu = \nu' \in \Lambda_{\zeta_e}^\circ$ and $a_{\mu\mu} \neq 0$. Since $(W_\xi^\mu : L_\xi^\mu) = 1$, we then also conclude that $a_{\mu\mu} = 1$.

(c) Since $\dim L_{\zeta_e}^\mu = \sum_{\nu \in \Lambda_\xi^\circ} a_{\mu\nu} \dim L_\xi^\nu \geq a_{\mu\mu} \dim L_\xi^\mu$, this follows from (b). \square

The observation that Λ_ξ° equals $\Lambda_{\zeta_e}^\circ$ once we know that these two sets have the same cardinality was first made by Jacon [27, Theorem 3.3] (in a slightly different context).

Theorem 3.3 (Geck–Rouquier [21, 5.4], [13, 3.2]). *Assume that $e\ell$ does not divide any degree of W . Then $|\text{Irr}(\mathcal{H}_k(W, \xi))| = |\text{Irr}(\mathcal{H}_\mathbb{C}(W, \zeta_e))|$.*

Actually, using some explicit computations for W of exceptional type and the results of Ariki–Mathas [2] for W of classical type, one can show that the above conclusion holds under the single assumption that ℓ is a good prime; see [13]. However, we do not need this stronger result here.

Remark 3.4. The significance of the assumption on ℓ in Theorem 3.3 is as follows. One easily checks that if $f \geq 2$ is such that $\Phi_f(\xi) = 0$ then $f = e\ell^i$ for some $i \geq 0$ (see, for example, [13, 3.1]). Hence, assuming that $e\ell$ does not divide any degree of W , we have the following implication for any $f \geq 2$:

$$\Phi_f(\xi) = 0 \text{ and } \Phi_f \text{ divides } P_W \quad \Rightarrow \quad f = e.$$

Conjecture 3.5 (General version of James' conjecture). *Recall our standing assumption that $e < \infty$ and $\text{char}(k) = \ell > 0$ where ℓ is a good prime for W . Assume also that ℓ*

does not divide any degree of W . Then the decomposition matrix D_ξ only depends on e . More precisely, the adjustment matrix A_ξ^e is the identity matrix or, in other words:

$$(J) \quad (W_\xi^\lambda : L_\xi^\mu) = (W_{\zeta_e}^\lambda : L_{\zeta_e}^\mu) \quad \text{for all } \lambda \in \Lambda \text{ and } \mu \in \Lambda_\xi^\circ = \Lambda_{\zeta_e}^\circ.$$

(Note that we do know that $\Lambda_\xi^\circ = \Lambda_{\zeta_e}^\circ$ by Theorem 3.3 and Lemma 3.2.)

Using the factorisation in Remark 3.1 and Lemma 3.2, the above conjecture can be reformulated as follows.

Corollary 3.6 (Alternative version of James' Conjecture). *Condition (J) in Conjecture 3.5 holds if and only if $\dim \text{rad}(\phi_\xi^\lambda) = \dim \text{rad}(\phi_{\zeta_e}^\lambda)$ for all $\lambda \in \Lambda$.*

Thus, in order to verify James' conjecture, it is sufficient to determine the ranks of the Gram matrices of the bilinear forms ϕ^λ for various specialisations. Recall from Section 2 that the entries of these Gram matrices are certain structure constants of \mathcal{H} with respect to its cellular basis, and these can be expressed in terms of the structure constants of the Kazhdan–Lusztig basis of \mathcal{H} . These in turn can be computed in principle (using recursive formulae), but note that this is only feasible for algebras of small rank. In Section 4 and [19], we will see how this problem can be solved effectively.

Proposition 3.7 (See also [8, Prop. 5.5] and [11, 2.7]). *There exists a bound N , depending only on W , such that condition (J) in Conjecture 3.5 holds for all $\ell > N$.*

Proof. We introduce the following notation. Given any matrix M with entries in A , we denote by M_ξ the matrix obtained by applying θ to all entries of M . Similarly, we define M_{ζ_e} via the map θ_e ; the entries of M_{ζ_e} will lie in $R[\zeta_{2e}]$. Finally, if N is a matrix with entries in $R[\zeta_{2e}]$, we denote by \bar{N} the matrix obtained by applying the map $\alpha \mapsto \bar{\alpha}$ to all entries of N (see Remark 3.1). With this notation, we have $M_\xi = \bar{M}_{\zeta_e}$ for any matrix M with entries in A .

Now fix $e \geq 2$ and $\lambda \in \Lambda$. Let G^λ be the Gram matrix of ϕ^λ ; this is a matrix with entries in $\mathbb{Z}[v, v^{-1}]$. With the above notation, we have $G_\xi^\lambda = \bar{G}_{\zeta_e}^\lambda$. This already implies that $\text{rank}(G_\xi^\lambda) \leq r := \text{rank}(G_{\zeta_e}^\lambda)$. We can find an $r \times r$ -submatrix G of G^λ such that $\det(G_{\zeta_e}^\lambda) \neq 0$. Now $\det(G_{\zeta_e}^\lambda)$ is an algebraic integer in the ring $\mathbb{Z}[\zeta_{2e}]$; its norm will be a non-zero rational integer. If ℓ does not divide that integer, we have

$$\det(G_\xi) = \det(\bar{G}_{\zeta_e}) = \overline{\det(G_{\zeta_e})} \neq 0.$$

So $r = \text{rank}(G_\xi^\lambda) = \text{rank}(G_{\zeta_e}^\lambda)$ for ℓ “large enough”. Hence, since Λ is a finite set, there is global bound N such that $\text{rank}(G_\xi^\lambda) = \text{rank}(G_{\zeta_e}^\lambda)$ for all $\lambda \in \Lambda$ and all $\ell > N$. Hence, by Corollary 3.6, the conclusion of James' conjecture holds for all $\ell > N$. \square

Note that the above proof actually provides a method for finding N , assuming that the Gram matrices G^λ are explicitly known.

Recall from Section 2 the definition of the Brauer graph of \mathcal{H} with respect to $\theta: A \rightarrow k$; its connected components are called ξ -blocks. Similarly, we define the Brauer graph of \mathcal{H} with respect to $\theta_e: A \rightarrow \mathbb{C}$. Its connected components are called ζ_e -blocks.

Definition 3.8. Given $\lambda \in \Lambda$, we set

$$\delta_\lambda := \max\{i \geq 0 \mid \Phi_e^i \text{ divides } \mathbf{c}_\lambda \text{ in } \mathbb{Q}[u]\}.$$

This number is called the Φ_e -defect of λ (or of E^λ).

Proposition 3.9 (Geck [8, 7.4 and 7.6]). *Assume that $e\ell$ does not divide any degree of W . Then the following hold.*

- (a) *The ξ -blocks of \mathcal{H} coincide with the ζ_e -blocks of \mathcal{H} .*
- (b) *If E^λ and E^μ belong to the same ξ -block, then $\delta_\lambda = \delta_\mu$.*

The above result shows that all irreducible representations in a given ξ -block of \mathcal{H} have the same Φ_e -defect, which will be called the Φ_e -defect of the block. Note that the only known proof of Proposition 3.9(b) relies on an interpretation of D_{ζ_e} in the modular representation theory of a finite group of Lie type with Weyl group W , and on known results on heights of characters in blocks of finite groups with abelian defect groups.

We can now state the main result of this article and its sequel [19].

Theorem 3.10. *Recall our standing assumption that $e < \infty$ and $\text{char}(k) = \ell > 0$ where ℓ is a good prime for W . Assume now that W is of exceptional type and that $e\ell$ does not divide any degree of W . Then James's conjecture holds for \mathcal{H} . More precisely, let Λ_1 be a ξ -block of Λ . By Proposition 3.9, Λ_1 has a well-defined Φ_e -defect, δ say.*

- (a) *If $\delta = 0$, then $\Lambda_1 = \{\lambda\}$ is a singleton set; we have $W_\xi^\lambda = L_\xi^\lambda$ and $W_{\zeta_e}^\lambda = L_{\zeta_e}^\lambda$.*
- (b) *If $\delta = 1$, then the following hold:*
 - (i) *We have $\mathbf{a}_\lambda \neq \mathbf{a}_{\lambda'}$ for any $\lambda \neq \lambda'$ in Λ_1 . Thus, we have a unique labelling $\Lambda_1 = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ such that $\mathbf{a}_{\lambda_1} < \mathbf{a}_{\lambda_2} < \dots < \mathbf{a}_{\lambda_n}$.*
 - (ii) *With the labelling in (i), we have $\Lambda_{1,\xi}^\circ = \{\lambda_1, \dots, \lambda_{n-1}\}$ and*

$$(W_\xi^{\lambda_i} : L_\xi^{\lambda_j}) = (W_{\zeta_e}^{\lambda_i} : L_{\zeta_e}^{\lambda_j}) = \begin{cases} 1 & \text{if } i = j \text{ or } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

- (c) *If $\delta \geq 2$, then $\Lambda_{1,\xi}^\circ$ and $\dim L_\xi^\mu$ for $\mu \in \Lambda_{1,\xi}^\circ$ are given by Tables 1 and 2.*

Remark 3.11. The ζ_e -blocks (together with their defect) of Iwahori–Hecke algebras of exceptional type are explicitly described in [20, App. F]. We have verified all the statements of Theorem 3.10 using an actual implementation of the algorithms presented in Section 4, and their refinements in [19]. Some of these statements are known to hold by theoretical arguments. More precisely:

- The statement in (a) follows from a general result about blocks of defect 0 in symmetric algebras; see [20, 7.5.11].
- The statement about $D_{\zeta_e,1}$ in (b) is proved, using general arguments, by a combination of [8, §10], [12, §4], [22, 4.4]. In [8, §10] it is also shown that these statements apply to D_ξ , if ℓ does not divide the order of W .

Note also that, once James' Conjecture is established (in the form of Corollary 3.6), the complete decomposition matrices can be easily determined: it is sufficient to compute them for *one* specialisation $\theta: A \rightarrow k$ where $\text{char}(k) = \ell$ is a good prime and $e\ell$ does not divide any degree of W . For the types F_4 , E_6 , E_7 , these matrices were known before and can be found in [18], [9], [10], [36]; for type E_8 , see [19].

4. Constructing the invariant bilinear form

We have seen in Proposition 3.7 that James' conjecture can be verified once we have constructed the Gram matrices of the invariant bilinear forms on the cell modules W^λ . If \mathcal{H} is not too large, we could actually do this by explicitly working out a cellular basis as

The sets $\Lambda_{\zeta_e}^\circ$ for type F_4, E_6, E_7

$F_4, e = 2$	$F_4, e = 2$	$F_4, e = 3$	$F_4, e = 3$	$F_4, e = 4$	$F_4, e = 6$
1 ₁ 0 1	4 ₂ 1 4	1 ₁ 0 1	4 ₂ 1 4	1 ₁ 0 1	1 ₀ 0 1
2 ₁ 1 2		2 ₁ 1 1	8 ₁ 3 4	4 ₂ 1 4	2 ₁ 1 2
2 ₃ 1 2		2 ₃ 1 1	8 ₃ 3 4	9 ₁ 2 4	2 ₃ 1 2
9 ₁ 2 5		4 ₁ 4 1	16 ₁ 4 4	6 ₁ 4 1	8 ₁ 3 5
				12 ₁ 4 4	8 ₃ 3 5

$E_6, e = 2$	$E_6, e = 3$	$E_6, e = 4$	$E_6, e = 6$
1_p 0 1	1_p 0 1	1_p 0 1	1_p 0 1
6_p 1 6	6_p 1 5	6_p 1 6	6_p 1 6
20_p 2 14	20_p 2 14	15_p 3 15	20_p 2 13
15_q 3 14	15_p 3 10	15_q 3 8	15_q 3 14
30_p 3 10	15_q 3 1	81_p 6 60	30_p 3 11
60_p 5 46	30_p 3 25	10_s 7 1	60_p 5 32
	64_p 4 10	80_s 7 6	24_p 6 11
	60_p 5 5	90_s 7 15	60_s 7 14
	60_s 7 14		80_s 7 13
	80_s 7 25		$60'_p$ 11 1
			$30'_p$ 15 6

$E_7, e = 2$	$E_7, e = 4$	$E_7, e = 4$	$E_7, e = 3$	$E_7, e = 6$
1 _a 0 1	1 _a 0 1	27 _a 2 27	1 _a 0 1	1 _a 0 1
7' _a 1 6	56' _a 3 56	21 _a 3 21	21 _a 3 21	7' _a 1 7
27 _a 2 14	105 _b 6 48	35 _b 3 8	35 _b 3 34	21' _b 3 13
35 _b 3 14	210 _a 6 154	216' _a 8 168	120 _a 4 98	21 _a 3 21
105' _a 4 78	189 _a 8 35	210 _b 10 7	105 _b 6 7	35 _b 3 27
189' _b 5 56	405 _a 8 147	105 _c 12 84	168 _a 6 35	15' _a 4 14
315' _a 7 126	70 _a 16 21	378 _a 14 105	210 _b 6 91	105' _a 4 77
	315 _a 16 120		280 _b 7 14	105 _b 6 43
			210 _b 10 49	168 _a 6 43
			420 _a 10 196	210 _a 6 92
$E_7, e = 2$	$E_7, e = 4$	$E_7, e = 4$	$E_7, e = 3$	
56' _a 3 56	7' _a 1 7	21' _b 3 21	7' _a 1 7	70' _a 7 42
120 _a 4 64	15' _a 4 8	120 _a 4 120	21' _b 3 14	280' _a 7 90
280 _b 7 216	105' _a 4 105	189' _b 5 48	56' _a 3 49	315' _a 7 13
	189' _c 7 84	35' _a 7 35	15' _a 4 1	84 _a 10 14
	280 _b 7 168	70' _a 7 1	105' _a 4 35	210 _b 10 27
	378' _a 9 21	315' _a 7 147	70' _a 7 21	420 _a 10 92
	210' _b 13 27	336 _a 13 154	280' _a 7 196	210' _b 13 1
		405' _a 15 56	336' _a 10 91	420' _a 13 77
			512 _a 11 98	280 _a 16 21
			84' _a 13 34	315 _a 16 7

Each table corresponds to a block of defect ≥ 2 . The first column specifies the set $\Lambda_{\zeta_e}^\circ$, the second column contains \mathbf{a}_μ and the third column contains $\dim L_{\zeta_e}^\mu$ for $\mu \in \Lambda_{\zeta_e}^\circ$.

Table 2

The sets $\Lambda_{\zeta_e}^\circ$ for type E_8

$E_8, e = 2$	$E_8, e = 4$	$E_8, e = 3$	$E_8, e = 6$	$E_8, e = 5$	$E_8, e = 8$
1_x 0 1	8_z 1 8	1_x 0 1	1_x 0 1	1_x 0 1	1_x 0 1
8_z 1 8	560_z 5 560	35_x 2 35	8_z 1 8	28_x 3 28	35_x 2 34
35_x 2 27	1344_x 7 784	28_x 3 28	35_x 2 35	84_x 3 83	160_z 4 160
84_x 3 48	840_z 10 56	84_x 3 48	28_x 3 28	567_x 6 539	567_x 6 373
50_x 4 42	1400_z 10 832	50_x 4 1	84_x 3 40	1344_x 7 722	175_x 8 174
210_x 4 202	4536_z 13 2360	210_x 4 147	50_x 4 41	972_x 10 166	1400_x 8 992
560_z 5 246	$4200'_z$ 21 1008	300_x 6 70	210_x 4 210	2268_x 10 1729	1575_x 8 1042
700_x 6 126	$2240'_x$ 28 1400	700_x 6 518	560_z 5 279	4096_z 11 1078	525_x 12 152
1400_z 7 792		1344_x 7 497	300_x 6 225	168_y 16 1	2835_x 14 1668
1050_x 8 651	$E_8, e = 4$	175_x 8 28	700_x 6 86	1134_y 16 28	6075_x 14 3516
1400_x 8 378	28_x 3 28	350_x 8 322	56_z 7 56	2688_y 16 722	2016_w 16 174
4200_x 12 1863	160_z 4 160	1050_x 8 35	448_z 7 85	4536_y 16 1729	5600_w 16 1042
	300_x 6 300	1400_x 8 1225	1400_z 7 489	$4096'_z$ 26 539	7168_w 16 992
	972_x 10 512	2240_x 10 322	175_x 8 85	$972'_x$ 30 83	$2835'_x$ 22 1
$E_8, e = 2$	840_x 12 28	4096_z 11 1036	350_x 8 266		$6075'_x$ 22 373
112_z 3 112	700_{xx} 13 512	4200_x 12 147	1050_x 8 660		$1400'_x$ 32 34
160_z 4 160	1344_w 16 160	700_{xx} 13 48	1400_x 8 259	$E_8, e = 10$	$1575'_x$ 32 160
400_z 6 288	$840'_x$ 24 300	3200_x 15 497	840_z 10 259	1_x 0 1	
1344_x 7 1184		4200_y 16 518	1400_z 10 40	8_z 1 8	
2240_x 10 1056	$E_8, e = 4$	4480_y 16 1225	840_x 12 41	28_x 3 28	$E_8, e = 12$
3360_z 12 2016	56_z 7 56		4200_x 12 1906	84_x 3 75	1_x 0 1
	1008_z 7 1008	$E_8, e = 3$	2100_x 13 1036	567_x 6 531	35_x 2 35
$E_8, e = 4$	1400_z 7 1400	8_z 1 8	2400_z 15 266	448_z 7 372	112_z 3 76
1_x 0 1	3240_z 9 832	112_z 3 104	4200_z 15 279	1008_z 7 449	50_x 4 50
35_x 2 34	2240_x 10 8	160_z 4 56	5600_z 15 489	1400_z 7 786	210_x 4 99
112_z 3 77	4200_z 15 2360	560_z 5 384	420_y 16 1	972_x 10 897	400_z 6 349
50_x 4 16	$3200'_x$ 21 784	400_z 6 8	1680_y 16 56	2268_x 10 502	1050_x 8 651
210_x 4 176	$4536'_z$ 23 560	448_z 7 56	4200_y 16 660	4536_z 13 2406	1400_x 8 974
567_x 6 280		1400_z 7 848	4480_y 16 8	1400_y 16 449	525_x 12 449
400_z 6 96	$E_8, e = 4$	840_z 10 448	4536_y 16 225	3150_y 16 372	3360_z 12 1386
175_x 8 1	84_x 3 84	1400_z 10 104	5670_y 16 28	4200_y 16 897	2800_z 13 1202
350_x 8 70	700_x 6 616	4096_x 11 1896	$4200'_x$ 24 35	4480_y 16 786	1400_y 16 99
1050_x 8 336	2268_x 10 1652	4200_z 15 384	$1400'_x$ 32 210	$4536'_z$ 23 75	2688_y 16 651
1575_x 8 946	4200_x 12 1848	5600_z 15 1896		$2268'_x$ 30 531	4536_y 16 974
525_x 12 168	2100_x 13 448	7168_w 16 848	$E_8, e = 6$	$448'_z$ 37 1	2100_y 20 449
3360_z 12 1654	2016_w 16 84		112_z 3 112	$1008'_z$ 37 28	$3360'_z$ 24 349
2800_z 13 1302	5600_w 16 1652		160_z 4 160	$1400'_z$ 37 8	$2800'_z$ 25 76
2835_x 14 34	$4200'_x$ 24 616		400_z 6 288		$1050'_x$ 32 50
6075_x 14 280			1344_x 7 1072		$1400'_x$ 32 1
3150_y 16 77			2240_x 10 768		$210'_x$ 52 35
4480_y 16 176			3360_z 12 2128		
5670_y 16 946			3200_x 15 2128		
			1344_w 16 288		
			7168_w 16 1072		
			$3360'_z$ 24 160		
			$2240'_x$ 28 112		

in [16, Exp. 4.3] (type B_2) or Example 2.7 (type G_2). Using computers, it would also be possible to carry out similar computations in type F_4 and, perhaps, type E_6 . However, this becomes totally unfeasible for type E_7 or E_8 , where we do have to explore alternative routes. The purpose of this section is to show how this can be done. Eventually, we will have to rely on computer calculations, but our aim is to develop a conceptual reduction of our problem where, at the end, standard programs like Parker's MeatAxe [38] and its variations can be applied. (See also Ringe's package [39] which comes with extensive documentation and a variety of additions to Parker's original programs.)

We keep the general setting of the previous section. Recall that \mathcal{H} is defined over the ring $A = R[v, v^{-1}]$ where $R \subseteq \mathbb{Q}$ consists of all fractions $a/b \in \mathbb{Q}$ such that $a \in \mathbb{Z}$ and $0 \neq b \in \mathbb{Z}$ is divisible by bad primes only. Let K be the field of fractions of A . If M is any A -module, we denote $M_K := K \otimes_A M$.

Let $e \geq 2$ and $\theta: A \rightarrow k$ a ring homomorphism into a field k ; let $\xi = \theta(u) \in k$. As before, if M is any A -module, we denote $M_\xi := k \otimes_A M$ where k is regarded as an A -module via θ . We say that θ is *e-regular* if $\text{char}(k) = \ell > 0$ is a good prime and $e\ell$ does not divide any degree of W . (These are precisely the conditions appearing in James' conjecture.) We will address the following three major issues which are sufficient for verifying that James' conjecture holds for a given algebra \mathcal{H} :

Problem 4.1. Let $e \geq 2$ be an integer which divides some degree of W .

- (a) For any $\lambda \in \Lambda$, construct an explicit model for W^λ , that is, an \mathcal{H} -module V^λ which is free of finite rank over A such that $V_K^\lambda \cong W_K^\lambda$. Determine Λ_ξ° and the decomposition matrix D_ξ for at least one e -regular specialisation $\theta: A \rightarrow k$.
- (b) Show that, for each $\lambda \in \Lambda_{\zeta_e}^\circ$, the model V^λ in (a) has the property that $V_\xi^\lambda \cong W_\xi^\lambda$ for any e -regular specialisation $\theta: A \rightarrow k$.
- (c) For any $\lambda \in \Lambda_{\zeta_e}^\circ$, determine the Gram matrix Q^λ of an invariant bilinear form on V^λ and show that $\text{rank}(Q_\xi^\lambda) = \text{rank}(G_\xi^\lambda)$ for any e -regular specialisation $\theta: A \rightarrow k$. Finally, compute $\text{rank}(Q_{\zeta_e}^\lambda)$ and find the *finite* set of prime numbers \mathcal{P}_e such that

$$\text{rank}(Q_\xi^\lambda) = \text{rank}(Q_{\zeta_e}^\lambda) \quad \text{if } \ell \notin \mathcal{P}_e.$$

4.1. Solving Problem 4.1(a)

Natural candidates for models for the cell representations of \mathcal{H} are the representations afforded by W -graphs. In fact, Gyoja [24] has shown that every irreducible representation of \mathcal{H}_K is afforded by a W -graph. We recall:

Definition 4.2 (Kazhdan–Lusztig [29]). A W -graph for \mathcal{H} consists of the following data:

- (a) a set X together with a map I which assigns to each $x \in X$ a set $I(x) \subseteq S$;
- (b) a collection of elements $\mu_{x,y} \in \mathbb{Z}$, where $x, y \in X$, $x \neq y$.

These data are subject to the following requirements. Let V be a free A -module with a basis $\{e_y \mid y \in X\}$. For each $s \in S$, define an A -linear map $\sigma_s: V \rightarrow V$ by

$$\begin{aligned} \sigma_s(e_y) &= v^2 e_y + \sum_{\substack{x \in X \\ s \in I(x)}} v \mu_{x,y} e_x & \text{if } s \notin I(y), \\ \sigma_s(e_y) &= -e_y & \text{if } s \in I(y). \end{aligned}$$

Then we require that the assignment $T_s \mapsto \sigma_s$ defines a representation of \mathcal{H} .

Thus, in a representation afforded by a W -graph, each generator T_s ($s \in S$) of \mathcal{H} is represented by a matrix of a particularly simple form. Recently, Howlett and Yin [26], [40] explicitly constructed W -graphs for all irreducible representations for Iwahori–Hecke algebras of type E_7 , E_8 . In combination with earlier results of Naruse [37] on types F_4 and E_6 , we now have W -graphs for all irreducible representations of algebras of exceptional type. These W -graphs are electronically accessible through Michel’s development version [35] of the computer algebra system CHEVIE [17]. Thus, we do have a collection of explicitly given \mathcal{H} -modules

$$\{V^\lambda \mid \lambda \in \Lambda\}$$

such that each V^λ is free of finite rank over A and $V_K^\lambda \cong E_v^\lambda \cong W_K^\lambda$.

Now let $\theta: A \rightarrow k$ be an e -regular specialisation. Using the CHOP function in Ringe’s version [39] of the MeatAxe, we can decompose each V_ξ^λ into its irreducible constituents. Thus, we obtain:

- $\text{Irr}(\mathcal{H}_k(W, \xi)) = \{M_1, \dots, M_r\}$ and
- the decomposition numbers $(V_\xi^\lambda : M_i)$ for $\lambda \in \Lambda$ and $1 \leq i \leq r$.

Note that, by Remark 2.8, we have $(W_\xi^\lambda : M_i) = (V_\xi^\lambda : M_i)$ for all $\lambda \in \Lambda$ and $1 \leq i \leq r$. The relations (Δ) in Section 2 immediately imply the following “identification result”:

Lemma 4.3. *Let $i \in \{1, \dots, r\}$. Then the unique $\mu \in \Lambda_\xi^\circ$ such that $M_i = L_\xi^\mu$ is determined by the conditions that $(W_\xi^\mu : M_i) = 1$ and*

$$\mathbf{a}_\mu \leq \mathbf{a}_\lambda \quad \text{for all } \lambda \in \Lambda \text{ such that } (W_\xi^\lambda : M_i) \neq 0.$$

By Theorem 3.3 and Lemma 3.2, we have $\Lambda_\xi^\circ = \Lambda_{\zeta_e}^\circ$. Thus, we are able to determine the sets $\Lambda_{\zeta_e}^\circ$ for any $e \geq 2$. This already yields the information contained in the first columns in Table 1 and 2.

4.2. Solving Problem 4.1(b)

Let us fix $e \geq 2$ and an element $\lambda \in \Lambda_{\zeta_e}^\circ$. As discussed above, we have an \mathcal{H} -module V^λ such that $W_K^\lambda \cong V_K^\lambda$. Now let $\theta: A \rightarrow k$ be any e -regular specialisation. In general, without any further knowledge about V^λ , we cannot expect that we also have $W_\xi^\lambda \cong V_\xi^\lambda$. The following result gives a precise condition for when this is the case.

Proposition 4.4. *Assume that there exists some e -regular specialisation $\theta_0: A \rightarrow k_0$ such that $V_{\xi_0}^\lambda$ (where $\xi_0 = \theta_0(u)$) has a unique maximal submodule U^λ , and we have $V_{\xi_0}^\lambda/U^\lambda \cong L_{\xi_0}^\lambda$. Then $V_{\zeta_e}^\lambda \cong W_{\zeta_e}^\lambda$ and $V_\xi^\lambda \cong W_\xi^\lambda$ for any e -regular specialisation $\theta: A \rightarrow k$.*

Proof. The module W^λ has a standard basis $\{C_\mathfrak{s} \mid \mathfrak{s} \in M(\lambda)\}$; let $\rho^\lambda: \mathcal{H} \rightarrow M_{d_\lambda}(A)$ be the corresponding matrix representation. The module V^λ also has a standard basis, arising from the underlying W -graph; let $\sigma^\lambda: \mathcal{H} \rightarrow M_{d_\lambda}(A)$ be the corresponding matrix representation. Since $V_K^\lambda \cong W_K^\lambda$, there exists an invertible matrix $P^\lambda \in M_{d_\lambda}(K)$ such that

$$\rho^\lambda(T_w)P^\lambda = P^\lambda\sigma^\lambda(T_w) \quad \text{for all } w \in W.$$

Multiplying P^λ by a suitable scalar, we may assume without loss of generality that

- all entries of P^λ lie in $\mathbb{Z}[v]$ and
- the greatest common divisor of all non-zero entries of P^λ is 1.

(Here we use the fact that R was chosen to be contained in \mathbb{Q} .) These conditions uniquely determine P^λ up to a sign. Let $\delta := \det(P^\lambda) \neq 0$. We need to obtain some more precise information about the irreducible factors of δ . Let us write

$$\delta = m f_1 f_2 \cdots f_r \quad \text{where } 0 \neq m \in \mathbb{Z} \text{ and } f_1, \dots, f_r \in \mathbb{Z}[v] \setminus \mathbb{Z} \text{ are irreducible.}$$

First we claim that m is divisible by bad primes only. Indeed, let p be a prime number which is good for W . Then p generates a prime ideal in R ; let $F = \mathbb{F}_p(v)$. We obtain a specialisation $\alpha: A \rightarrow F$ by reducing the coefficients of polynomials in A modulo p . We have a corresponding specialised algebra $\mathcal{H}_F(W, u)$. Since $\alpha(P_W) \neq 0$, we conclude that $\mathcal{H}_F(W, u)$ is semisimple and the specialised cell modules W_F^λ are all irreducible; see Proposition 2.4. Now note that not all entries of P^λ are divisible by p . Hence, reducing the entries of P^λ modulo p , we obtain a non-zero matrix defining a non-trivial module homomorphism $V_F^\lambda \rightarrow W_F^\lambda$. Since W_F^λ is irreducible and $\dim W_F^\lambda = \dim V_F^\lambda$, this homomorphism must be an isomorphism. Consequently, P^λ is invertible modulo p and so p cannot divide m .

A similar argument shows that each f_i divides $P_W(v^2)$. Indeed, assume that $f \in \mathbb{Z}[v]$ is a non-constant irreducible polynomial which does not divide $P_W(v^2)$. Then we have a canonical ring homomorphism $\beta: A \rightarrow F$ where $F = \mathbb{Q}[v]/(f)$. Again, the corresponding specialised algebra $\mathcal{H}_F(W, \theta(u))$ is semisimple since $\beta(P_W) \neq 0$. Arguing as above, we conclude that f does not divide $\det(P^\lambda)$. Thus, each f_i must divide $P_W(v^2)$.

Now consider the specialisation $\theta_e: A \rightarrow \mathbb{C}$ which sends v to ζ_{2e} . We can actually regard this as a map with image in $\mathbb{Q}(\zeta_{2e})$ and work with $\mathbb{Q}(\zeta_{2e})$ instead of \mathbb{C} as base field. Thus, $W_{\zeta_e}^\lambda$ and $V_{\zeta_e}^\lambda$ can be regarded as $\mathbb{Q}(\zeta_{2e})$ -vectorspaces and modules for the specialised algebra $\mathcal{H}_{\mathbb{Q}(\zeta_{2e})}(W, \zeta_e)$. Let \mathcal{O} be a discrete valuation ring as in Remark 3.1 with respect to the specialisation θ_0 ; we have a corresponding decomposition map

$$d_{\xi_0}^e: R_0(\mathcal{H}_{\mathbb{Q}(\zeta_{2e})}(W, \zeta_e)) \rightarrow R_0(\mathcal{H}_{k_0}(W, \xi_0)).$$

Once again, since the greatest common divisor of all its entries is 1, the matrix P^λ induces a non-trivial module homomorphism $V_{\zeta_e}^\lambda \rightarrow W_{\zeta_e}^\lambda$. We claim that this also is an isomorphism. To prove this, let $M \subseteq V_{\zeta_e}^\lambda$ be the kernel of the map $V_{\zeta_e}^\lambda \rightarrow W_{\zeta_e}^\lambda$; then M is a proper submodule of $V_{\zeta_e}^\lambda$. By a standard result (see [4, 23.7]), there exists a proper submodule $N \subseteq V_{\xi_0}^\lambda$ such that

$$d_{\xi_0}^e([M]) = [N] \quad \text{and} \quad d_{\xi_0}^e([V_{\zeta_e}^\lambda/M]) = [V_{\xi_0}^\lambda/N].$$

If $L_{\zeta_e}^\lambda$ were a composition factor of M , then $L_{\xi_0}^\lambda$ would be a composition factor of N by Lemma 3.2(b). But then, by our assumption on $V_{\xi_0}^\lambda$ and since $N \subseteq U$, the simple module $L_{\xi_0}^\lambda$ would appear at least twice as a composition factor of $V_{\xi_0}^\lambda$, which is absurd. So we conclude that $L_{\zeta_e}^\lambda$ is not a composition factor of M . Hence, $L_{\zeta_e}^\lambda$ will be a composition factor of the image of the map $V_{\zeta_e}^\lambda \rightarrow W_{\zeta_e}^\lambda$. But, by [23, Prop. 3.2], $L_{\zeta_e}^\lambda$ is a simple quotient of $W_{\zeta_e}^\lambda$, the kernel of the canonical map $W_{\zeta_e}^\lambda \rightarrow L_{\zeta_e}^\lambda$ is the unique maximal submodule of $W_{\zeta_e}^\lambda$, and $L_{\zeta_e}^\lambda$ is not a composition factor of that kernel. So we conclude that the map $V_{\zeta_e}^\lambda \rightarrow W_{\zeta_e}^\lambda$ is surjective and, hence, an isomorphism. It follows that δ is not divisible by $\Phi_{2e}(v)$. If e is odd, we can also consider the specialisation $\theta'_e: A \rightarrow \mathbb{C}$ sending v to

$\zeta_e^{(e+1)/2}$ (the other square root of ζ_e , which is a root of $\Phi_e(v)$). Then a similar argument shows that δ is not divisible by $\Phi_e(v)$. Thus, we have reached the following conclusions:

- m is divisible by bad primes only;
- each f_i divides $P_W(v^2)$;
- each f_i is coprime to $\Phi_e(v^2)$.

We can now complete the proof as follows. Let $\theta: A \rightarrow k$ be any e -regular specialisation. Assume that $\theta(\delta) = 0$. Since the characteristic of k is a good prime, we must have $\theta(f_i) = 0$ for some $i \in \{1, \dots, r\}$. Since each f_i divides $P_W(v^2)$, there exists some $d \geq 2$ such that $\Phi_d(v^2)$ divides $P_W(v^2)$ and f_i divides $\Phi_d(v^2)$. By Remark 3.4, we conclude that $d = e$. Thus, we see that f_i divides $\Phi_e(v^2)$, a contradiction. Hence, our assumption was wrong and so we do have $\theta(\delta) \neq 0$. Thus, we have shown that P^λ induces an isomorphism $V_\xi^\lambda \xrightarrow{\sim} W_\xi^\lambda$. \square

Let $\theta_0: A \rightarrow k_0$ be a specialisation as in Proposition 4.4. Using the MKSUB function in Ringe's version [39] of the MeatAxe (see also [33]), we can determine the complete submodule lattice of $V_{\xi_0}^\lambda$. Using the CHOP function and Lemma 4.3 as discussed in the previous subsection, we can identify the various irreducible constituents of $V_{\xi_0}^\lambda$ and check that the assumption of Proposition 4.4 is satisfied. Thus, Problem 4.1(b) is solved.

It might actually be true that W^λ and V^λ are isomorphic as \mathcal{H} -modules, but we have not been able to prove this. We would like to state this as a conjecture:

Conjecture 4.5. *Assume that, for each $\lambda \in \Lambda$, we are given a W -graph affording an \mathcal{H} -module V^λ such that $V_K^\lambda \cong E_v^\lambda$. Then the cellular basis in Theorem 2.2 can be chosen such that $W^\lambda \cong V^\lambda$ for all $\lambda \in \Lambda$.*

4.3. Solving Problem 4.1(c)

Let $\lambda \in \lambda_{\zeta_e}^\circ$ and G^λ be the Gram matrix of the invariant bilinear form ϕ^λ with respect to the standard basis of W^λ . Instead of W^λ , we now consider the module V^λ and assume that the hypotheses of Proposition 4.4 are satisfied. Thus, we have $V_{\zeta_e}^\lambda \cong W_{\zeta_e}^\lambda$ and $V_\xi^\lambda \cong W_\xi^\lambda$ for any e -regular specialisation $\theta: A \rightarrow k$.

Let $\sigma^\lambda: \mathcal{H} \rightarrow M_{d_\lambda}(A)$ be the matrix representation afforded by V^λ with respect to the standard basis arising from the underlying W -graph. Our task now is to find some non-zero matrix $Q^\lambda \in M_{d_\lambda}(A)$ such that

$$(*) \quad Q^\lambda \cdot \sigma^\lambda(T_s) = \sigma^\lambda(T_s)^{\text{tr}} \cdot Q^\lambda \quad \text{for all } s \in S.$$

Note that $(*)$ implies that $Q^\lambda \cdot \sigma^\lambda(T_{w^{-1}}) = \sigma^\lambda(T_w)^{\text{tr}} \cdot Q^\lambda$ for all $w \in W$. So any solution to $(*)$ is the Gram matrix of an invariant bilinear form on V^λ . Multiplying Q^λ by a suitable scalar, we may assume without loss of generality that

- all entries of Q^λ lie in $\mathbb{Z}[v]$ and
- the greatest common divisor of all non-zero entries of Q^λ is 1.

Note that, by Schur's Lemma, any two matrices satisfying $(*)$ are scalar multiples of each other. Hence, the above conditions uniquely determine Q^λ up to a sign.

Lemma 4.6. *Assume that Q^λ is a solution to $(*)$ satisfying the above conditions. Then*

$$\text{rank}(Q_{\zeta_e}) = \text{rank}(G_{\zeta_e}^\lambda) \quad \text{and} \quad \text{rank}(Q_\xi) = \text{rank}(G_\xi^\lambda)$$

for any e -regular specialisation $\theta: A \rightarrow k$.

Proof. We are assuming that $\lambda \in \Lambda_{\zeta_e}^\circ = \Lambda_\xi^\circ$, so we have $G_{\zeta_e}^\lambda \neq 0$ and $G_\xi^\lambda \neq 0$.

Now let P^λ be as in the proof of Proposition 4.4 and set $\tilde{Q}^\lambda := (P^\lambda)^{\text{tr}} G^\lambda P^\lambda$. Then \tilde{Q}^λ is a solution to $(*)$ and so there exists some $0 \neq \alpha \in K$ such that $\tilde{Q}^\lambda = \alpha Q^\lambda$. Since all three matrices G^λ , Q^λ and \tilde{Q}^λ have all their entries in $\mathbb{Z}[v, v^{-1}]$ and since the greatest common divisor of the entries of Q^λ is 1, we can conclude that $\alpha \in \mathbb{Z}[v, v^{-1}]$.

Now, in the proof of Proposition 4.4, we have actually seen that $P_{\zeta_e}^\lambda$ and P_ξ^λ are invertible. Since we also have $G_{\zeta_e}^\lambda \neq 0$ and $G_\xi^\lambda \neq 0$, it follows that

$$\text{rank}(\tilde{Q}_{\zeta_e}) = \text{rank}(G_{\zeta_e}^\lambda) > 0 \quad \text{and} \quad \text{rank}(\tilde{Q}_\xi) = \text{rank}(G_\xi^\lambda) > 0.$$

But then it also follows that $\theta_e(\alpha) \neq 0$ and $\theta(\alpha) \neq 0$. Hence, we have $\text{rank}(\tilde{Q}_{\zeta_e}) = \text{rank}(Q_{\zeta_e}^\lambda)$ and $\text{rank}(\tilde{Q}_\xi) = \text{rank}(Q_\xi^\lambda)$, and this yields the desired statement. \square

It remains to show how a solution to $(*)$ can actually be computed. Note that $(*)$ constitutes a system of $|S|d_\lambda^2$ linear equations for the d_λ^2 entries of Q^λ . If d_λ is not too large, this can be solved directly. However, in type E_8 , we have $d_\lambda = 7168$ for some λ , and our system of linear equations simply becomes too large. In such cases, different techniques are required which are based on the following result:

Theorem 4.7 (Benson–Curtis; see [20, §6.3]). *Each $E^\mu \in \text{Irr}(W)$ is of parabolic type, that is, there exists a subset $I \subseteq S$ such that the restriction of E^μ to the parabolic subgroup $W_I \subseteq W$ contains the trivial representation of W_I just once. A similar statement holds when “trivial representation” is replaced by “sign representation”.*

Now the main idea is as follows: Since the bijection $\text{Irr}(W) \leftrightarrow \text{Irr}(\mathcal{H}_K)$ arising from Tits’ Deformation Theorem is compatible with restriction to parabolic subgroups and subalgebras (see [20, 9.1.9]), the above result means that there exists a subset $I \subseteq S$ such that

$$\dim_K \left(\bigcap_{s \in I} \ker(\sigma_K^\lambda(T_s + T_1)) \right) = 1;$$

let $e_1 \in K^{d_\lambda}$ be a vector spanning this one-dimensional space. Similarly,

$$\dim_K \left(\bigcap_{s \in I} \ker(\sigma_K^\lambda(T_s + T_1)^{\text{tr}}) \right) = 1;$$

let $v_1 \in K^{d_\lambda}$ be a vector spanning this one-dimensional space. Now, since σ_K^λ is an irreducible representation of \mathcal{H}_K , there exist $w_2, \dots, w_{d_\lambda} \in W$ such that the vectors

$$e_1, \quad e_2 := \sigma^\lambda(T_{w_2})e_1, \quad e_3 := \sigma^\lambda(T_{w_3})e_1, \quad \dots, \quad e_{d_\lambda} := \sigma^\lambda(T_{w_{d_\lambda}})e_1,$$

form a basis of K^{d_λ} . Then the vectors

$$v_1, \quad v_2 := \sigma^\lambda(T_{w_2^{-1}})^{\text{tr}} v_1, \quad v_3 := \sigma^\lambda(T_{w_3^{-1}})^{\text{tr}} v_1, \quad \dots, \quad v_{d_\lambda} := \sigma^\lambda(T_{w_{d_\lambda}^{-1}})^{\text{tr}} v_1,$$

will also form a basis of K^{d_λ} . Hence, there exists a unique invertible matrix $\tilde{Q}^\lambda \in M_{d_\lambda}(K)$ such that $v_i = \tilde{Q}^\lambda e_i$ for $1 \leq i \leq d_\lambda$. Then $\tilde{Q}^\lambda \cdot \sigma^\lambda(T_w) \cdot (\tilde{Q}^\lambda)^{-1} = \sigma^\lambda(T_{w^{-1}})^{\text{tr}}$ for all $w \in W$ and so \tilde{Q}^λ is a solution to $(*)$. Multiplying by a suitable scalar, we obtain Q^λ .

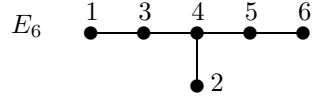
The above technique is known as the “standard base” algorithm; see the ZSB function of Ringe’s MeatAxe [39] and its description. In practice, we did not apply it to σ^λ itself

but to various specialisations into finite fields such that the specialised algebra remains semisimple. For each such specialisation, we use the ZSB function to find the Gram matrix of an invariant bilinear form. Using interpolation and modular techniques (Chinese Remainder), one can recover Q^λ from these specialisations.

Having computed Q^λ , we substitute $v \mapsto \sqrt[2e]{1}$ and determine the rank of the specialised matrix. Arguing as in the proof of Proposition 3.7, we find the *finite* set of prime numbers \mathcal{P}_e such that $\text{rank}(Q_\xi^\lambda) = \text{rank}(Q_{\zeta_e}^\lambda)$ if $\ell \notin \mathcal{P}_e$. See [19] for further details.

Remark 4.8. Assume we are in the above setting, where $I \subseteq S$ is a subset such that the restriction of E^λ to W_I contains the sign representation exactly once. Then, by the formulas in Definition 4.2, the vector e_1 can be taken to be contained in the standard basis of K^{d_λ} . Since $v_1 = \tilde{Q}^\lambda e_1$, we conclude that v_1 is a column of the matrix \tilde{Q}^λ . In other words, using Theorem 4.7, one column of the matrix \tilde{Q}^λ can be computed by simply determining the intersection of the kernels of the maps $\sigma_K^\lambda(T_s + T_1)^{\text{tr}}$ where s runs over the generators in I .

Example 4.9. In general, the matrix Q^λ is far from being sparse. We just give one example. Let W be of type E_6 with Dynkin diagram



Consider the unique 10-dimensional irreducible representation, which is denoted 10_s in [20, Table C.4]. By Naruse [37], a W -graph is given by Table 3. (The numbers inside a circle specify the subset $I(x)$; all $\mu_{x,y}$ are 0 or 1; we have an edge between x and y if and only if $\mu_{x,y} = 1$.) From this graph, we find that the basis vector with $I(x) = \{1, 2, 3, 5, 6\}$ spans the one-dimensional intersection of kernels considered above (in accordance with [20, Table C.4]). This basis vector labels the last row and column in the matrix of Q^{10_s} in Table 3.

In this case, the determination of the bound required by James' conjecture is very easy. By Table 1, we have $10_s \in \Lambda_{\zeta_4}^\circ$. If we specialise $v \mapsto \zeta_8$, we notice that $Q_{\zeta_4}^{10_s}$ has rank 1; all rows become equal to

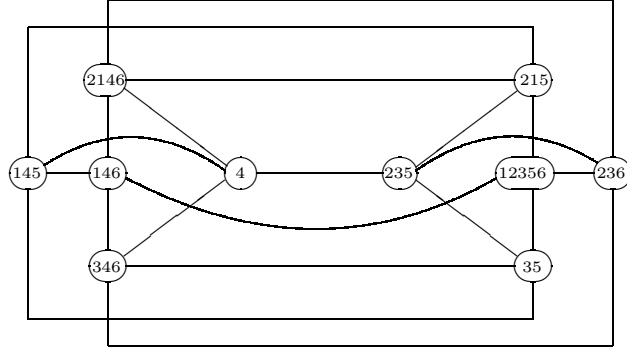
$$\left[-2 + 2\zeta_4, -2 + 2\zeta_4, -2 + 2\zeta_4, -2\zeta_8^3, -2 + 2\zeta_4, -2\zeta_8^3, -2 + 2\zeta_4, -2\zeta_8^3, -2\zeta_8^3, -2\zeta_8^3 \right]$$

We see that, if we specialise further into a field of characteristic $\ell > 0$, we will still obtain a matrix of rank 1 unless $\ell = 2$.

Remark 4.10. We have been able to systematically compute the matrices Q^λ (with coefficients in A) for all λ such that $d_\lambda \leq 2500$. For those λ in type E_8 where this wasn't feasible (at least not with the computer power available to us), we nevertheless managed to compute directly the specialized matrices $Q_{\zeta_e}^\lambda$ for all relevant values of e . Note that this is sufficient to find the finite set of prime numbers \mathcal{P}_e as above. (See [19] for details.) There is an on-going project to complete the determination of all "generic" matrices Q^λ and to create a data base for making them generally available.

References

Table 3
 W -graph and invariant bilinear form for the representation 10_s in type E_6



$$Q^{10_s} = \begin{bmatrix} v^6+3v^4+3v^2+1 & 2v^4+2v^2 & 2v^4+2v^2 & -v^5-2v^3-v & 2v^4+2v^2 & \\ 2v^4+2v^2 & v^6+3v^4+3v^2+1 & 2v^4+2v^2 & -v^5-2v^3-v & 2v^4+2v^2 & \\ 2v^4+2v^2 & 2v^4+2v^2 & v^6+3v^4+3v^2+1 & -v^5-2v^3-v & 2v^4+2v^2 & \\ -v^5-2v^3-v & -v^5-2v^3-v & -v^5-2v^3-v & v^6+2v^4+2v^2+1 & -v^5-2v^3-v & \\ 2v^4+2v^2 & 2v^4+2v^2 & 2v^4+2v^2 & -v^5-2v^3-v & v^6+3v^4+3v^2+1 & \\ -v^5-2v^3-v & -v^5-2v^3-v & -v^5-2v^3-v & v^4+v^2 & -2v^3 & \\ 2v^4+2v^2 & 2v^4+2v^2 & 2v^4+2v^2 & -2v^3 & 2v^4+2v^2 & \\ -v^5-2v^3-v & -v^5-2v^3-v & -2v^3 & v^4+v^2 & -v^5-2v^3-v & \\ -v^5-2v^3-v & -2v^3 & -v^5-2v^3-v & v^4+v^2 & -v^5-2v^3-v & \\ -2v^3 & -v^5-2v^3-v & -v^5-2v^3-v & v^4+v^2 & -v^5-2v^3-v & \\ \\ -v^5-2v^3-v & 2v^4+2v^2 & -v^5-2v^3-v & -v^5-2v^3-v & -2v^3 & \\ -v^5-2v^3-v & 2v^4+2v^2 & -v^5-2v^3-v & -2v^3 & -v^5-2v^3-v & \\ -v^5-2v^3-v & 2v^4+2v^2 & -2v^3 & -v^5-2v^3-v & -v^5-2v^3-v & \\ v^4+v^2 & -2v^3 & v^4+v^2 & v^4+v^2 & v^4+v^2 & \\ -2v^3 & 2v^4+2v^2 & -v^5-2v^3-v & -v^5-2v^3-v & -v^5-2v^3-v & \\ v^6+2v^4+2v^2+1 & -v^5-2v^3-v & v^4+v^2 & v^4+v^2 & v^4+v^2 & \\ -v^5-2v^3-v & v^6+3v^4+3v^2+1 & -v^5-2v^3-v & -v^5-2v^3-v & -v^5-2v^3-v & \\ v^4+v^2 & -v^5-2v^3-v & v^6+2v^4+2v^2+1 & v^4+v^2 & v^4+v^2 & \\ v^4+v^2 & -v^5-2v^3-v & v^4+v^2 & v^6+2v^4+2v^2+1 & v^4+v^2 & \\ v^4+v^2 & -v^5-2v^3-v & v^4+v^2 & v^4+v^2 & v^6+2v^4+2v^2+1 & \end{bmatrix}$$

- [1] S. ARIKI, On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$, J. Math. Kyoto Univ. **36** (1996), 789–808.
- [2] S. ARIKI AND A. MATHAS, The number of simple modules of the Hecke algebras of type $G(r, 1, n)$, Math. Z. **233**, 601–623.
- [3] R. W. CARTER, Simple groups of Lie type, Wiley, New York, 1972.
- [4] C. W. CURTIS AND I. REINER, Methods of representation theory Vol. I and II, Wiley, New York,

- 1981 and 1987.
- [5] R. DIPPER AND G. D. JAMES, Representations of Hecke algebras of the general linear groups, *Proc. London Math. Soc.* **52** (1986), 20–52.
 - [6] W. FEIT, *The representation theory of finite groups*, North-Holland Publishing Company, Amsterdam, 1982.
 - [7] F. FUNG, On the topology of components of some Springer fibers and their relation to Kazhdan–Lusztig theory, *Adv. Math.* **178** (2003), 244–276.
 - [8] M. GECK, Brauer trees of Hecke algebras, *Comm. Algebra* **20** (1992), 2937–2973.
 - [9] M. GECK, The decomposition numbers of the Hecke algebra of type E_6 , *Math. Comp.* **61** (1993), 889–899.
 - [10] M. GECK, *Beiträge zur Darstellungstheorie von Iwahori-Hecke-Algebren*, Habilitationsschrift, Aachener Beiträge zur Mathematik **11**, Verlag der Augustinus Buchhandlung, Aachen, 1995.
 - [11] M. GECK, Representations of Hecke algebras at roots of unity (Séminaire Bourbaki, 50ème année, 1997–98, Exp. 836), *Astérisque* **252** (1998), 33–55.
 - [12] M. GECK, Kazhdan–Lusztig cells and decomposition numbers, *Represent. Theory* **2** (1998), 264–277 (electronic)
 - [13] M. GECK, On the number of simple modules of Iwahori–Hecke algebras of finite Weyl groups. *Bul. Stiit. Univ. Baia Mare, Ser. B* **16** (2000), 235–246.
 - [14] M. GECK, Modular representations of Hecke algebras. *In: Group representation theory* (EPFL, 2005; eds. M. Geck, D. Testerman and J. Thévenaz), Presses Polytechniques et Universitaires Romandes, EPFL-Press, 2006.
 - [15] M. GECK, Kazhdan–Lusztig cells and the Murphy basis, *Proc. London Math. Soc.* **93** (2006), 635–665.
 - [16] M. GECK, Hecke algebras of finite type are cellular, *Invent. Math.* **169** (2007), 501–517.
 - [17] M. GECK, G. HISS, F. LÜBECK, G. MALLE, AND G. PFEIFFER, CHEVIE—A system for computing and processing generic character tables, *Appl. Algebra Engrg. Comm. Comput.* **7** (1996), 175–210; electronically available at <http://www.math.rwth-aachen.de/~CHEVIE>
 - [18] M. GECK AND K. LUX, The decomposition numbers of the Hecke algebra of type F_4 , *Manuscripta Math.* **70** (1991), 285–306.
 - [19] M. GECK AND J. MÜLLER, James’ Conjecture for Hecke algebras of exceptional type, II, *in preparation*.
 - [20] M. GECK AND G. PFEIFFER, *Characters of finite Coxeter groups and Iwahori–Hecke algebras*, London Math. Soc. Monographs, New Series **21**, Oxford University Press, 2000.
 - [21] M. GECK AND R. ROUQUIER, Centers and simple modules for Iwahori–Hecke algebras, *Finite reductive groups, related structures and representations* (M. Cabanes, ed.), *Progress in Math.* **141**, Birkhäuser, Boston, 1997, pp. 251–272.
 - [22] M. GECK AND R. ROUQUIER, Filtrations on projective modules for Iwahori–Hecke algebras, *in: Modular Representation Theory of Finite Groups* (Charlottesville, VA, 1998; eds. M. J. Collins, B. J. Parshall and L. L. Scott), p. 211–221, Walter de Gruyter, Berlin 2001.
 - [23] J. J. GRAHAM AND G. I. LEHRER, Cellular algebras, *Invent. Math.* **123** (1996), 1–34.
 - [24] A. GYOJA, On the existence of a W -graph for an irreducible representation of a finite Coxeter group. *J. Algebra* **86** (1984), 422–438.
 - [25] A. GYOJA AND K. UNO, On the semisimplicity of Hecke algebras, *J. Math. Soc. Japan* **41** (1989), no. 1, 75–79.
 - [26] R. B. HOWLETT, W -graphs for the irreducible representations of the Hecke algebras of type E_7 and E_8 ; private communication with J. Michel (December 2003).
 - [27] N. JACON, Canonical basic sets for Hecke algebras, *Contemp. Math., Amer. Math. Soc.* **392** (2005), 33–41.
 - [28] G. D. JAMES, The decomposition matrices of $GL_n(q)$ for $n \leq 10$, *Proc. London Math. Soc.* **60** (1990), 225–265.
 - [29] D. KAZHDAN AND G. LUSZTIG, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.
 - [30] A. LASCOUX, B. LECLERC, J-Y. THIBON, Hecke algebras at roots of unity and crystal bases of quantum affine algebras. *Comm. Math. Phys.* **181** (1996), 205–263.
 - [31] G. LUSZTIG, Characters of reductive groups over a finite field, *Annals Math. Studies* **107**, Princeton University Press, 1984.

- [32] G. LUSZTIG, Hecke algebras with unequal parameters, CRM Monographs Ser. **18**, Amer. Math. Soc., Providence, RI, 2003.
- [33] K. LUX, J. MÜLLER AND M. RINGE, Peakword condensation and submodule lattices: an application of the MeatAxe, J. Symb. Comput. **17** (1994), 529–544.
- [34] T. P. McDONOUGH AND C. A. PALLIKAROS, On relations between the classical and the Kazhdan–Lusztig representations of symmetric groups and associated Hecke algebras, J. Pure and Applied Algebra **203** (2005), 133–144.
- [35] J. MICHEL, Homepage of the development version of the GAP part of CHEVIE; see <http://www.institut.math.jussieu.fr/~jmichel/chevie/chevie.html>
- [36] J. MÜLLER, Zerlegungszahlen für generische Iwahori-Hecke-Algebren von exzeptionellem Typ, Dissertation, RWTH Aachen, 1995.
- [37] H. NARUSE, W -graphs for the irreducible representations of the Iwahori–Hecke algebras of type F_4 and E_6 ; private communication (January and July, 1998).
- [38] R. A. PARKER, The computer calculation of modular characters (the Meat-Axe), *In*: M. D. Atkinson (ed.), *Computational group theory*, Academic Press, London, 1984.
- [39] M. RINGE, The C MeatAxe release 2.4, Lehrstuhl D für Mathematik, RWTH Aachen, 2004; electronically available at <http://www.math.rwth-aachen.de/~MTX>
- [40] Y. YIN, W -Graph representations for Coxeter groups and Hecke algebras, Ph.D. thesis, University of Sydney, 2004.